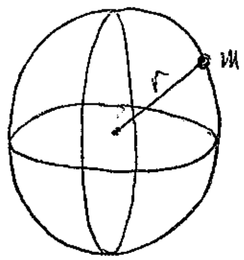
Rotation in 3-D (particle on a sphere)

A little more complicated but similar problem to the particle on a ring is the particle on a sphere problem. Here we assume that the particle mass m freely moves over the surface of a sphere radius r.



since the particle moves on the surface of the sphere unhindered, the potential energy is zero

$$V(r) = 0$$

Thus, the total energy E equals to the kinetic energy, K.

From quantum mechanics,

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \quad (38,1)$$

Again, it is natural to solve this problem in the spherical polar coordinate

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) \quad (38,2)$$

This looks <u>scary</u>!!! However, since r is constant and $I = mr^2$, we can simplify it to

$$-\frac{\hbar^2}{2m}\nabla^2\psi = -\frac{\hbar^2}{2I}\left\{\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)\right\} = E\psi \quad (38,3)$$

This is a partial differential equation in the two angular variables θ and ϕ. We can solve this equation using the separation of variable technique.

First we assume that Ψ can be written as

$$\Psi = \Theta \cdot \Phi \tag{39.1}$$

where Θ is a function of only θ and Φ is a function of only ϕ.

Second, substitute Ψ into eq. (38.3), we obtain two partial differential equations (one is for θ and the other for ϕ), which we can then solve separately.

We will not solve these eqs. here, but rather examine the properties of the solutions.

i) <u>wave functions</u>: The wavefunctions have the form of the spherical harmonic, $Y_{l,m}(\theta,\phi)$, and depend on two quantum numbers, l and m.

$$\boxed{\begin{aligned} l &= 0, 1, 2, \ldots \\ m &= -l, -l+1, \ldots, 0, 1, \ldots l \end{aligned}}$$

l	m	$Y_{l,m}(\theta,\phi)$
0	0	$\left(\frac{1}{4\pi}\right)^{1/2}$
1	0	$\left(3/4\pi\right)^{1/2}\cos\theta$
1	± 1	$\mp\left(3/8\pi\right)^{1/2}\sin\theta\, e^{\pm i\phi}$
⋮	⋮	⋮

2/ Energy

$$\boxed{E = \ell(\ell+1)\frac{\hbar^2}{2I}} \qquad \ell = 0, 1, 2 \ldots \qquad (40.1)$$

Thus, energy is quantized and depends only on the ℓ quantum number. Since for each ℓ, there are $(2\ell+1)$ distint values of m, and hence $(2\ell+1)$ distint $Y_{\ell,m}(\theta,\phi)$ corresponding to the same energy level, a energy level of quantum number ℓ is $(2\ell+1)$ fold degenerate.

$\overline{}$ $m=-2$ $m=-1$ $m=0$ $m=1$ $m=2$ $\ell = 2$ $E_2 = 3\frac{\hbar^2}{I}$

$m=-1$ $m=0$ $m=1$ $\ell = 1$ $E_1 = \frac{\hbar^2}{I}$

$m=0$ $\ell = 0$ $E_0 = 0$

Angular momentum

Recall from classical mechanics, the energy of a rotating particle is related to its angular momentum, J, by

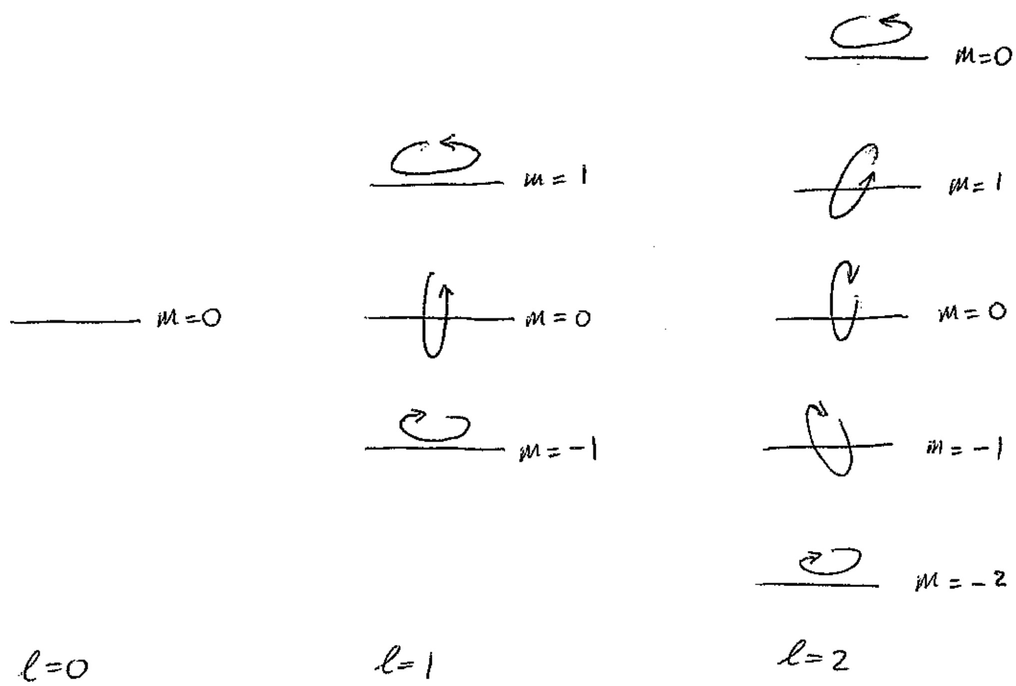
$$E = \frac{J^2}{2I} \qquad (40.2)$$

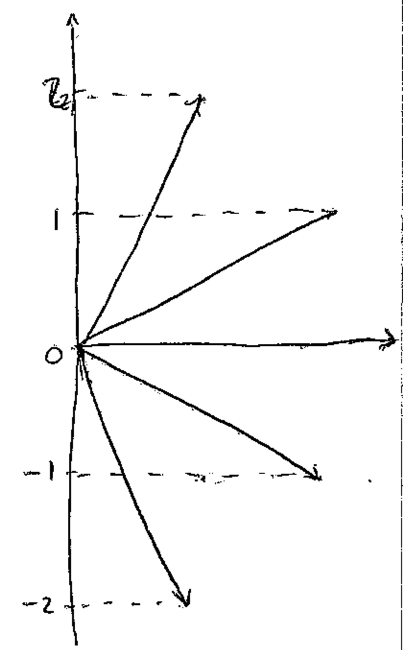
Compare eq. (40.2) to (40.1), we obtain

$$\boxed{J = \sqrt{\ell(\ell+1)}\,\hbar} \qquad \ell = 0, 1, 2, \ldots \qquad (40.3)$$

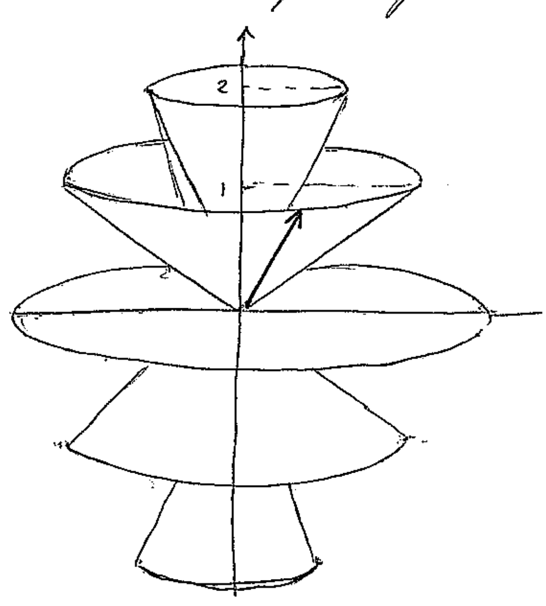
So angular momentum is quantized. Earlier, we already showed that angular momentum about the z-axis is also quantized with discrete values

$$\boxed{J_z = m\hbar} \qquad m = 0, \pm 1, \pm 2, \cdots, \pm \ell$$



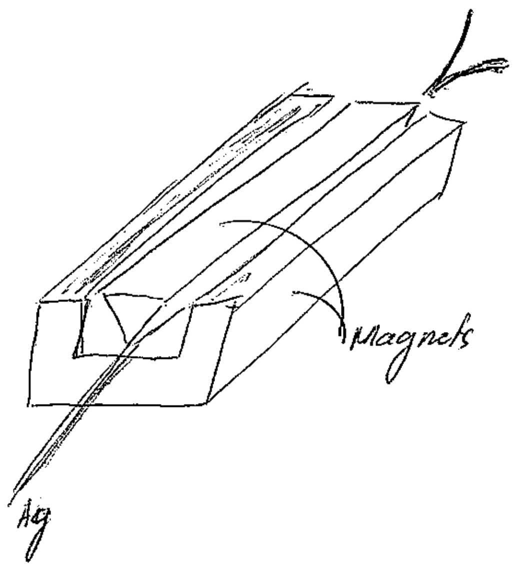


Up to now, we've seen examples of energy quantization. For the particle on a sphere, we show that the angular momentum is quantized. This indicates that the plane of rotation of the particle can take only a discrete range of orientations. This quantum mechanical result is called space orientation. However, due to the uncertainty principle, we can specify one component of the angular momentum. We choose to specify the z-component, L_z, thus it leaves the x- and y-components completely undefined.



The state of the angular momentum with a definite z-component can be represented by a vector with its tip on any point on the mouth of the cone which has the specified constant z-projection.

The space quantization phenomenon was first confirmed by the experiment of Stern and Gerlach in 1921, which was based on the idea that a rotating charge body would generate an intrinsic magnetic field which can interact with the external applied field.



The experiment was done by shooting a beam of Ag atoms through an inhomogeneous magnetic field. The applied field deflects the stream of Ag atoms into two sharp bands.

Since the angular momentum of Ag atom is quantized, and each angular momentum with quantum number l gives rise to $(2l+1)$ orientation. The experimental observation indicates $l = \frac{1}{2}$. This contradicts with our earlier proof that l must be an integer.

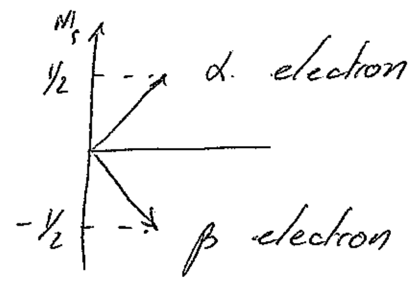
WHY?

Electron spin

Electron posses an intrinsic motion about its own axis called spin which has the magnitude of the spin angular momentum to be $\sqrt{s(s+1)}\,\hbar$ and $(2s+1)$ possible orientation with the z-compoments, m_s have values

$$m_s = s, s-1, \ldots, -s$$

For electron, it has only one allowed s value, and $s = \frac{1}{2}$



Thus, the Stern & Gerlach's observation can be explained if we suppose that each Ag atom possesses an effective angular momentum due ~~the~~ to the spin of a single electron.

Electron also known as fermion particle has ~~sp~~ an intrinsic spin-$\frac{1}{2}$, but a boson particle such as photon has an intrinsic spin-1.